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NOTE ON THE ASYMPTOTICS OF SMALL SELF-OSCILLATIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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A method of computing small self-oscillations of the systems of ordinary differential equations which is sufficiently simple for practical applications, is given. The coefficients of the corresponding expansions are obtained from the linear algebraic equations. A detailed analysis of the problem on the origination of self-oscillations from the state of equilibrium was carried out in [1] for the systems of differential equations on a plane. This was developed further in [2 – 6]. In [7 – 11] the problem on the origination of self-oscillations was studied for the systems with lag. The self-oscillations are normally determined by expanding them in fractional powers of a small parameter. The difficulties which arise from the lack of uniqueness in determining the expansion coefficients, can be overcome [11] by separating the coefficients of the initial expansions into two groups; the coefficients of the first group are then determined with the accuracy of up to certain parameters which enter the equations in a nonlinear manner, and the equations determining the second group of the coefficients are solved with the aid of the refined parameters of the first group.

Below we give another method of obtaining an asymptotic expansion of self-oscillations and of their periods, in which the coefficients of the asymptotic expansion are determined consecutively and uniquely from the recurrent linear relations.

1. Let R^n be an n-dimensional Euclidean space and $B^n(r) = \{x \in R^n : \|x\| < r\}$. Let the function $F(x, \varepsilon)$ be defined in $B^n(r) \times [0, \varepsilon_0]$ $(\varepsilon_0 > 0)$ and analytic, and let it assume values in R^n. We shall consider the following system of ordinary differential equations:

$$dx / dt = F(x, \varepsilon) \tag{1.1}$$

We assume that $F(0, \varepsilon) \equiv 0$, consequently the system is in a null state of equilibrium for all ε. Let also the matrix $A_1 = F_x(0, 0)$ have a pair of pure imaginary characteristic roots $\pm i$ and let the remaining characteristic roots lie in the left complex half-plane. We denote by e_1 and e_2 the vectors for which $Ae_1 = e_2$, $Ae_2 = -e_1$. Then, for the conjugate matrix A_1^* vectors g_1 and g_2 can be found such that $A_1^* g_1 = -g_2$, $A_1^* g_2 = g_1$ and $g_i, e_i = \delta_{ij}$ $(i, j = 1, 2)$. We assume that

$$a_1 = 1/2\,[(g_1, B_1 e_1) + (g_2, B_1 e_2)] \neq 0, \quad B_1 = \frac{\partial}{\partial \varepsilon} F_x(0, 0) \tag{1.2}$$

Then for small ε, the matrix $A_1 + \varepsilon B_1$ has a pair of characteristic roots $z_{1,2}(\varepsilon) = \alpha(\varepsilon) \pm i\beta(\varepsilon)$, where $\alpha(0) = 0$, $\beta(0) = 1$ and $\alpha'(0) = a_1$. Thus, at small $\varepsilon > 0$, the stability of the null state of equilibrium is determined by the sign of a_1.

When $\varepsilon = 0$, the linear terms no longer determine the stability of the null state of equilibrium. We shall assume that the null solution is either asymptotically stable, or unstable. Following [6] we shall say that at $\varepsilon = 0$ there is a "change in stability" of the null state of equilibrium if for $\varepsilon = 0$ the null state of equilibrium of the system (1.1) is asymptotically stable (unstable), while for small $\varepsilon > 0$, it is unstable (asymptotically stable). This definition differs somewhat from the usual one (see e.g. [1]), however we find it useful when considering wider classes of systems with a small parameter accompanying the derivative, and of the systems with a lagging argument. It was shown in [6, 9] that the change in stability of the state of equilibrium is accompanied by the onset of self-oscillations.

2. We shall seek an asymptotic expansion of this self-oscillation in the form of a series in integral powers of some auxiliary parameter c which is geometrically equivalent to the length of the projection of the initial condition x_0 of some self-oscillation on e_1. We shall also expand the small parameter ε into a special series in powers of c. Thus we choose, in fact, the parameter ε in such a manner that the system (1.1) has a self-oscillation, the projection c of which on the vector e_1 is given. When ε is varied monotonously, the parameter c varies in the same manner. Let

$$F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + \dots$$

We set three formal series

$$x(\tau, c) = cx_1(\tau) + c^2 x_2(\tau) + \dots \tag{2.1}$$
$$h(c) = 1 + h_1 c + h_2 c^2 + \dots, \quad \gamma(c) = \gamma_1 c + \gamma_2 c^2 + \dots$$

Here $x_i(\tau)$ are 2π-periodic vector functions which assume values in R^n and are, as yet, unknown, h_i and γ_i are numbers which are also unknown and c is a small positive parameter. In what follows, we shall choose the functions $x_i(\tau)$ so that

$$(g_1, x_1(0) - 1 = (g_1, x_{k+1}(0)) = (g_2, x_k(0)) = 0, \quad (k = 1, 2, \dots) \tag{2.2}$$

The series (2.1) and the integer $m > 2$ are obtained from the formal identity

$$\frac{dx(\tau, c)}{d\tau} \equiv h(c)\{F_0[x(\tau, c)] + \gamma^{m-1}(c) F_1[x(\tau, c)] + \gamma^{2(m-1)}(c) F_2[x(\tau, c)] + \dots\} \tag{2.3}$$

by equating the coefficients of like powers of c. Clearly $x_1, \dots, x_{m-1}$ and $h_1, \dots, h_{m-2}$. are determined from the formal identity

$$dx(\tau, c)/d\tau \equiv h(c) F_0[x(\tau, c)] \tag{2.4}$$

Let $F_0(x) = A_1 x + A_2(x, x) + A_3(x, x, x) + \dots$, where $A_s(y_1, \dots, y_s)$ are polylinear symmetric operators acting from $R^n \times \dots \times R^n$ into R^n. Then for x_1 we have the following equation

$$dx_1/d\tau = A_1 x_1$$

The above equation has a two-parameter set of 2π-periodic solutions $x = c_1\varphi_1(\tau) + c_2\varphi_2(\tau)$, and it can be assumed that $\varphi_1(0) = e_1$, $\varphi_2(0) = e_2$. From (2.2) it follows that $x_1 = \varphi_1$. Let us assume that $x_1, \dots, x_p$ and $h_1, \dots, h_{p-1}$ are successively determined. Then for determination of x_{p+1} and h_p we have

$$dx_{p+1}/d\tau = A_1 x_{p+1} + h_p A_1 x_1 + X_{p+1}(x_1, \dots, x_p, h_1, \dots, h_{p-1}) \tag{2.5}$$

In order for the above equation to have a 2π-periodic solution, it is necessary and sufficient that

$$\alpha_p = \frac{1}{2\pi}\int_0^{2\pi} (h_p A_1 x_1 + X_{p+1}, \psi_1)\, d\tau = 0 \tag{2.6}$$

$$\beta_p = \frac{1}{2\pi}\int_0^{2\pi}(h_p A_1 x_1 + X_{p+1}, \psi_2)\,d\tau = 0 \tag{2.7}$$

where ψ_1 and ψ_2 are 2π-periodic solutions of the conjugate system $y = -A_1^*, y$, satisfying the conditions $\psi_1(0) = g_1$ and $\psi_2(0) = g_2$. Since $A_1\varphi_1 = -\varphi_2$ and $(\varphi_2, \psi_2) \equiv 1$, it follows that h_p is uniquely obtained from (2.7). The relation (2.6) may or may not hold. When it does hold, Eq.(2.5) has a 2π-periodic solution which is determined uniquely from the conditions (2.2). Otherwise we have $\alpha_p \neq 0$, and the sign of α_p determines the stability of the null state of equilibrium of (1.1) with $\varepsilon = 0$ (see [12]).

Let p be the first number for which $\alpha_p \neq 0$. We set $m = p + 1$ and extend the procedure of determining $x_{p+1}, x_{p+2}, \ldots; h_p, h_{p+1}, \ldots, \gamma_1$ and γ_2 beginning from this number and using the formal identity (2.3).

Let us write $F_1(x)$ in the form

$$F_1(x) = B_1 x + B_2(x, x) + \ldots$$

where $B_s(y_1, \ldots, y_s)$ denote the polylinear symmetric operators acting from $R^n \times \ldots \times R^n$ into R^n.

From (2.4) we obtain the following equation for the 2π-periodic function $x_{p+1}(\tau)$:

$$dx_{p+1}/d\tau = A_1 x_{p+1} + h_p A_1 x_1 + \gamma_1^p B_1 x_1 + X_{p+1}(x_1, \ldots, x_p, h_1, \ldots, h_{p-1}) \tag{2.8}$$

Simple calculations show that

$$\int_0^{2\pi}(B_1\varphi_1, \psi_1)\,d\tau = 2\pi a_1, \qquad \int_0^{2\pi}(A_1\varphi_1, \psi_1)\,d\tau = 0$$

where a_1 is determined from (1.2). Therefore Eq. (2.8) has a 2π-periodic solution if

$$a_1\gamma_1^p + \frac{1}{2\pi}\int_0^{2\pi}(X_{p+1}, \psi_1)\,d\tau = 0$$

$$h_p + \frac{1}{2\pi}\int_0^{2\pi}[(X_{p+1}, \psi_2) + (B_1\varphi_1, \psi_2)\gamma_1^p]\,d\tau = 0$$

The first of the above relations yields $\gamma_1 = (-\alpha_p / a_1)^{1/p}$, and the second one yields h_p. Taking into account (2.2) we obtain from (2.8) a unique x_{p+1}. Let us assume that $x_{p+1}, \ldots, x_{p+s-1}; h_p, \ldots, h_{p+s-2}; \gamma_1, \ldots, \gamma_{s-1}$ have all been consecutively determined. A simple calculation shows that for the determination of x_{p+s}, h_{p+s-1} and γ_s we obtain the following expression:

$$\begin{aligned} dx_{p+s}/d\tau = A_1 x_{p+s} + h_{p+s-1}A_1 x_1 + p\gamma_1^{p-1}\gamma_s B_1 x_1 + \\ X_{p+s}(x_1, \ldots, x_{p+s-1}; h_1, \ldots, h_{p+s-2}; \gamma_1, \ldots, \gamma_{s-1}) \end{aligned} \tag{2.9}$$

Since $\gamma_1 \neq 0$, the condition of existence of a 2π-periodic solution of (2.9) yields γ_s and h_{p+s-1}, and hence x_{p+s} (uniquely).

Thus the determination of x_i, h_{i-1} and γ_{i-p} can be extended to any integral value of $k > 0$.

3. Let us now have some definite value for $\varepsilon > 0$, and attempt to determine the self-oscillation with the accuracy of up to $\varepsilon^{k/p}$ (k is an integer). Using the procedure given above we construct x_i, h_{i-1} and γ_{i-p} $(i = 1, \ldots, k)$ and set

$$x_k(\tau, c) = \sum_{i=1}^{k} c^i x_i(\tau), \quad h_k(c) = 1 + \sum_{i=1}^{k} c^i h_i, \quad \gamma_k(c) = \sum_{i=1}^{k-p} c^i \gamma_i$$

Next, we use the method of undetermined coefficients to find the function $c = c(\varepsilon^{1/p})$ as the solution of the equation $\gamma_k(c) = \varepsilon^{1/k}$. Finally we set

$$x_k^*(t, \varepsilon) = x_k[t / h(c(\varepsilon^{1/p})), \ c(\varepsilon^{1/p})]$$

This yields an approximate self-oscillation with the accuracy of up to $\varepsilon^{k/p}$.

To justify this procedure we note that for the system (1.1) there exists a unique self-oscillation $x(t, \varepsilon)$ which can be expanded together with its period $T(\varepsilon)$ into an asymptotic expansion in the fractional powers of the parameter ε

$$x(t, \varepsilon) = x_1(t)\varepsilon^{1/p} + x_2(t)\varepsilon^{2/p} + \ldots, \qquad T(\varepsilon) = 2\pi + T_1\varepsilon^{1/p} + T_2\varepsilon^{2/p} + \ldots$$

on each finite segment $0 \leqslant t \leqslant \Delta$. Moreover, this solution satisfies the conditions $(g_1, x(0, \varepsilon)) = (-a_1 / \alpha_p)\varepsilon^{1/p} + o(\varepsilon^{1/p})$, $(g_2, x(0, \varepsilon)) = 0$ (this has actually been stated in [6]). If we now set $c = (g_1, x(0, \varepsilon))$, then $\varepsilon^{1/p} = \gamma_1 c + \gamma_2 c^2 + \ldots$ and the solution $x(t, \varepsilon)$ and period $T(\varepsilon)$ can be uniquely determined by the series

$$x(\tau, c) = x_1^*(\tau)c + x_2(\tau)c^2 + \ldots$$

$$T(c) = 2\pi(1 + h_1 c + h_2 c^2 + \ldots), \qquad \tau = t / (1 + h_1 c + \ldots)$$

where $x_i(\tau)$ are 2π-periodic functions satisfying the conditions (2.2). The above expansions were obtained using the method described above.

Example. We consider the equation

$$x'' + \varepsilon x' + x - \beta x^2 + \alpha x'^3 = 0 \tag{3.1}$$

where the prime denotes the time derivative, and find a small self-oscillation with the accuracy of $o(\varepsilon)$. To do this, we make the substitution $t = H\tau$ and seek $x(\tau)$ in the form

$$x(\tau) = x_1(\tau)c + x_2(\tau)c^2 + x_3(\varepsilon)c^3 + \ldots$$

where $x_i(\tau)$ are 2π-periodic functions. Let us set

$$H = 1 + h_2 c^2 + h_3 c^3 + \ldots, \qquad \varepsilon = (\gamma_1 c + \gamma_2 c^2 + \ldots)^{m-1}$$

where h_i and γ_i are coefficients, as yet unknown. A simple calculation shows that

$$Hx^{\cdot\cdot} = x_1^{\cdot\cdot}c + x_2^{\cdot\cdot}c^2 + (x_3^{\cdot\cdot} + h_2 x_1^{\cdot\cdot})c^3 + (x_4^{\cdot\cdot} + h_2 x_2^{\cdot\cdot} + h_3 x_1^{\cdot\cdot})c^4 + \ldots$$

$$Hx^{\cdot} = cx_1^{\cdot} + c^2 x_2^{\cdot} + \ldots$$

$$H^3 x = x_1 c + x_2 c^2 + (3h_2 x_1 + x_3)c^3 + (x_4 + 3h_2 x_2 + 3h_3 x_1)c^4 + \ldots$$

$$H^3 x^2 = x_1^2 c^2 + 2x_1 x_2 c^3 + (3h_2 x_1^2 + 2x_1 x_3 + x_2^2)c^4 + \ldots$$

$$x^{\cdot 3} = x_1^{\cdot 3}c^3 + 3x_1^{\cdot 2}x_2^{\cdot}c^4 + \ldots$$

where dot superscripts denote derivatives with respect to τ. We substitute the above expressions into the equation obtained from (3.1) by replacing t, and equate the coefficients of like powers of c. For x_1 we obtain the equation $x_1^{\cdot\cdot} + x_1 = 0$ from which it follows that $x_1 = \cos\tau$. Further, $x^{\cdot\cdot}_2 + x_2 = \beta\cos^2\tau$. From this we obtain, taking into account the condition that $x_2(0) = x_2^{\cdot}(0) = 0$,

$$x_2(\tau) = {}^1/_6\,\beta(3 - 2\cos\tau - \cos 2\tau)$$

Calculation shows that in the present case we have $m = 3$.

Equating the coefficients of c cubed, we obtain the following equation for x_3:

$$x_3'' + x_3 = \gamma_1^2 \sin\tau - 2h_2\cos\tau + \beta/3^2 \cos\tau\,(3 - 2\cos\tau - \cos 2\tau) + \alpha\sin^3\tau$$

The condition of existence of a 2π-periodic solution to this equation yields

$$\gamma_1 = {}^1/_2 \sqrt{-3\alpha}, \quad h_2 = {}^5/_{12}\beta^2$$

It can be shown that x_3 has the form

$$x_3 = a_0 + a_1\cos\tau + b_1\sin\tau + a_2\cos 2\tau + a_3\cos 3\tau + b_3\sin 3\tau$$

Finally, equating the coefficients of c^4 we obtain

$$x_4'' + x_4 = -h_2x_2'' - h_3x_1'' - \gamma_1^2x_2' - 2\gamma_1\gamma_2x_1' - 3h_2x_2 - 3h_3x_1 + \beta\,(3h_2x_1^3 + 2x_1x_3 + x_2^2) - 3\alpha x_1'^2x_2'$$

The condition of orthogonality of the function $\sin\tau$ and of the right-hand side of the last equation, yields $-{}^1/_6\gamma_1^2 + \gamma_1\gamma_2 = {}^3/_8\alpha\beta$. From this it follows that $\gamma_2 = -{}^1/_6\beta\sqrt{-3\alpha}$. From the relation $\varepsilon^{1/2} = \gamma_1c + \gamma_2c^2 + \ldots$ it follows that

$$c = \sigma_1\varepsilon^{1/2} + \sigma_2\varepsilon + \ldots, \quad \sigma_1 = -2\sqrt{-3\alpha}\,/\,3\alpha, \quad \sigma_2 = -4\beta\,/\,9\alpha$$

therefore we have

$$x = -\frac{2\varepsilon^{1/2}\cos\tau}{\sqrt{-3\alpha}} - \frac{2\beta\varepsilon}{3\alpha}\left(1 - \frac{1}{3}\cos 2\tau\right) + \ldots, \quad \tau = \frac{t}{1 - 5\beta^2\varepsilon/9\alpha + \ldots}$$

The procedure shown here can be generalized to embrace the systems of singularly perturbed equations [6]. It can also be used to compute the self-oscillations in the systems with lagging arguments.

REFERENCES

1. Andronov, A. A. and Leontovich, E. A., Certain cases of the dependence of the limit cycles on the parameters. Uch. zap. Gor'kovsk. Univ. № 6, 1939.
2. Hopf, E., Berlin. Sähs. Acad. Wiss., Leipzig. Math. Phis., kl., 94, Nr. 19, 1972.
3. Neimark, Iu. I., Point mapping methods in the theory of nonlinear oscillations. Izv. vuzov, Radiofizika, №№ 5–6, 1958.
4. Brushlinskaia, N. N., Indefinite integration of a system of differential equations in the region containing a singularity and a limit cycle. Dokl. Akad. Nauk, SSSR, Vol. 139, № 1, 1961.
5. Chafee, N., The bifurcation of one or more closed orbits from an equilibrium point of an autonomous differential system. J. Diff, Equat., Vol. 4, 1968.
6. Strygin, V. V., "Change in stabilities" and bifurcation of small self-oscillations of a system of differential equations with a small parameter accompanying the higher derivative. Dokl. Akad. Nauk SSSR, Vol. 199, № 1, 1971.
7. Chefi, N., Problem of bifurcation for the differential-difference equations with lag. Tr. V Mezhdunar. konf. po nelineinym kolebaniiam, Vol. 1, Kiev, Izd. Inst. mat., Akad. Nauk UkrSSR, 1970.
8. Strygin, V. V., Bifurcation of self-oscillations in the system of functional equations. Tr. matem. fak. Voronezh. Univ., № 4, 1971.
9. Strygin, V. V., Bifurcation of small self-oscillations in singularly perturbed systems with lag. Tr. NII matem. Voronezh. Univ., № 5, 1972.
10. Neimark, Iu. I. and Fishmen, L. Z., Structure of the phase space of the quasi-linear differential equations of neutral type. Izv. vuzov Radiofizika, Vol. 15, № 11, 1972.

11. Kolesov, Iu. S., Computation of self-oscillations of nonlinear differential equations with aftereffect, branching from the null state of equilibrium. Vestn. Iaroslav. Univ., №2, 1973.
12. Malkin, I. G., Theory of Stability of Motion. Moscow, "Nauka", 1966.

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STRESS TENSOR AND AVERAGING IN MECHANICS OF CONTINUOUS MEDIA

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The problem of determining the mean stress and the other macrovariables originates upon passing from the equations of motion which are valid in the microscale, to the macroscopic equations which describe the motion of continuous media (such as a turbulized fluid, an elastic medium with microdefects, the suspension of gas bubbles or solid particles in a fluid, etc.). The mean value of the stress tensor over a volume was introduced in the monograph [1], and precisely this quantity was used in the governing relations to compute the Einstein viscosity of suspensions. Moreover, some effective representation in terms of integrals over surfaces [1] was used in specific calculations of these means with respect to the volume. Later, Batchelor [2], and some other authors after him [3], used precisely these means with respect to the volume as the stresses in the macroequations of motion by assuming the equivalence between the average with respect to a volume and with respect to a surface. Hence, in particular, the absolute symmetry of the macrostress tensor follows in the above-mentioned cases.

In this paper it is shown that the average of the microstress tensor and the microflux of the momenta with respect to the volume according to the rule in [1] determines only some symmetric part of the complete macrostress tensor. For the simple case of a viscous fluid moving inhomogeneously over a microlevel, this mean of the tensor with respect to the volume is related linearly to the mean strain rates. Moreover, the representation used in [1] permits clarification of the essential difference between the mean stresses with respect to the volume and with respect to the surfaces, in the general case.

The method of integrating the microequations with respect to the vloume [4 – 6] naturally results in the appearance of stresses in the macroequations, which are the means with respect to the differential macroareas. It is essential that the macrostress tensory is hence generally nonsymmetric although the equations of motion in the microscale correspond to symmetric continuum mechanics. It is this consideration which permitted the development of the continuum equations of motion of a suspension, which reflects the effect of nonequilibrium intrinsic rotation of the suspended particles [7], and the case of a turbulized fluid with anisotropies of eddy character is set in conformity to the nonzero antisymmetric part of the Reynolds stresses [8].